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Singular Nonlinear Boundary Value Problems Arising in Boundary Layer Theory

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The singular nonlinear boundary value problem

$$\begin{cases} w''(t) = -\lambda\left(\dfrac{1-t^2}{w(t)}\right)' - \dfrac{t}{w(t)}, & 0 \le t < 1, \\ w'(0)w(0) = -\lambda, \quad w(1) = 0, \end{cases}$$

arises in the boundary layer theory. We prove in this paper that the problem has a unique positive solution for each fixed $\lambda \geq 0$ and there is no positive solution for $\lambda \leq -1/2$. © 1999 Academic Press

1. INTRODUCTION

The third-order nonlinear ordinary differential equation for $f(\eta)$,

$$f''' + ff'' + \lambda(1 - f'^2) = 0, \qquad 0 < \eta < +\infty, \tag{1.1}$$

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with boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(+\infty) = 1, \tag{1.2}$$

is of great importance in the boundary layer theory in fluid mechanics. Equation (1.1) was first deduced by Falkner and Skan [4]. The cases $\lambda = 0$ and $\lambda = \frac{1}{2}$ of (1.1) are often called the Blasius and Homann differential equations, respectively. Many authors have investigated the boundary value problem (1.1)–(1.2) by using numerical and analytical methods. For details, see [5, pp. 519–537 , 7, pp. 149–151] and the references therein. In this paper, we present a new approach to study the problem (1.1)–(1.2) and provide some new information about a solution to the problem.

The introduction of a new independent variable t and a new unknown function $w(t)$ can transform the problem (1.1)–(1.2) into a singular nonlinear boundary value problem of second order. Assume that $f(\eta)$ is a solution to the boundary value problem (1.1)–(1.2) and $f''(\eta) > 0$ for all $\eta \geq 0$. Then $\eta = g(t)$, the inverse function to $t = f'(\eta)$, exists and strictly increasing in $[0, 1)$, $g(0) = 0$, $g(1 - 0) = \infty$, and

$$t \equiv f'(g(t)) \quad \text{for all } t \in [0, 1). \tag{1.3}$$

Differentiating (1.3) with respect to t yields

$$w(t) := f''(g(t)) = \frac{1}{g'(t)}, \qquad 0 \leq t < 1. \tag{1.4}$$

Here and henceforth a prime denotes differentiation with respect to the variable t or the variable η in each case. Substituting $\eta = g(t)$ into (1.1), we obtain

$$w'(t)w(t) + f(g(t))w(t) + \lambda(1 - t^2) = 0, \qquad 0 \leq t < 1. \tag{1.5}$$

Here we have used (1.3), (1.4) and the fact that

$$w'(t) = f'''(g(t))g'(t) = \frac{f'''(g(t))}{w(t)}, \qquad 0 \leq t < 1.$$

Dividing (1.5) by $w(t)$ and then differentiating it with respect to t, we get

$$w''(t) = -\lambda\left(\frac{1 - t^2}{w(t)}\right)' - \frac{t}{w(t)}, \qquad 0 \leq t < 1. \tag{1.6}$$

From (1.2), (1.5) and $g(1 - 0) = +\infty$, it follows that

$$w'(0)w(0) = -\lambda, \qquad w(1) = 0. \tag{1.7}$$

Problem (1.6)–(1.7) is singular at $t = 1$. The particular case $\lambda = 0$ of the problem has been studied by many authors. For detailed analysis of the case or a general form of the case, one is referred to [2, 3, 6, 8, 9]. As far as we know, the singular nonlinear boundary value problem (1.6)–(1.7) has not been studied yet when $\lambda \neq 0$.

Problem (1.6)–(1.7) can be used as a model example to study a singular nonlinear boundary value problem.

In Section 2 we will prove that for each fixed $\lambda \geq 0$ the problem (1.6)–(1.7) has a positive solution $w(t; \lambda)$ and no positive solution exists for $\lambda \leq -\frac{1}{2}$. Our arguments for the existence and uniqueness of the $w(t; \lambda)$ involve a perturbation technique, comparison principles and the Schauder fixed point theorem. It is unfortunate that our arguments are not able to deal with the case $\lambda \in (-\frac{1}{2}, 0)$ of (1.6)–(1.7) and hence cannot be applied to generalize the results to (1.1)–(1.2) in this range. There are some detailed statements to (1.1)–(1.2) for $\lambda < 0$ in [7].

In Section 3 we will construct a unique solution $f(\eta; \lambda)$ to the boundary value problem (1.1)–(1.2) by utilizing the $w(t; \lambda)$ and explore the asymptotic behavior of $f'(\eta; \lambda)$ as $\eta \to +\infty$.

2. PROBLEM (1.6)–(1.7)

The present section is the core of this paper in which we deal with the singular nonlinear boundary value problem (1.6)–(1.7).

A function $w(t)$ is said to be a positive solution to the problem (1.6)–(1.7), if

(i) $w(t) \in C[0,1] \cap C^2[0,1)$, $w(t) > 0$ for all $t \in [0,1)$ and

(ii) $w(t)$ satisfies (1.6) and (1.7).

It is easy to see that a function $w(t)$ is a positive solution to the problem (1.6)–(1.7) if and only if it satisfies (i) and solves the integral equation

$$w(t) = \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)\, ds}{w(s)} + (1-t)\int_0^t \frac{s\, ds}{w(s)}, \qquad 0 \leq t \leq 1. \tag{2.1}$$

From this we know that

$$w'(t) = -\frac{\lambda(1-t^2)}{w(t)} - \int_0^t \frac{s\, ds}{w(s)}, \qquad 0 \leq t < 1, \tag{2.2}$$

which shows that for each fixed $\lambda \geq 0$, the positive solution $w(t)$ is strictly decreasing on $[0,1]$.

According to (2.1) and (2.2), we can establish the following *a priori* estimates.

THEOREM 2.1. *Assume that* $\lambda \geq 0$ *and the problem* (1.6)–(1.7) *has a positive solution* $w(t;\lambda)$. *Then*

$$w(t;\lambda) \geq A(1-t) \quad \text{on} \quad [0,1], \qquad A := \sqrt{(4\lambda+1)/6}, \tag{2.3}$$

and

$$w(t;\lambda) \leq A^{-1}(1-t)[2\lambda+1-\log(1-t)] \quad \text{on} \quad [0,1]. \tag{2.4}$$

Proof. We first prove (2.3). Since $\lambda \geq 0$, it follows from (2.1) and (2.2) that

$$\begin{aligned} w(t;\lambda) &\geq \frac{1}{w(t;\lambda)} \int_t^1 (1-s)(\lambda+\lambda s+s)\, ds \\ &= \left[3(1-t)^2(\lambda+\lambda t+t) + (1-t)^3(\lambda+1)\right]/6w(t;\lambda) \\ &\geq (4\lambda+1)(1-t)^2/6w(t;\lambda), \qquad 0 \leq t < 1, \end{aligned}$$

i.e., (2.3) holds.

We now prove (2.4). From (2.1) and (2.3) it follows that

$$\begin{aligned} w(t;\lambda) &\leq \int_t^1 \frac{(1-s)(\lambda+\lambda s+s)\, ds}{A(1-s)} + (1-t)\int_0^t \frac{s\, ds}{A(1-s)} \\ &\leq \frac{1}{A}(1-t)[2\lambda+1-\log(1-t)], \qquad 0 \leq t \leq 1. \end{aligned}$$

This is exactly the desired result.

We now state and prove the first comparison principle.

THEOREM 2.2. *Let* $w(t;\lambda)$ *be a positive solution to the problem* (1.6)–(1.7) *with* $\lambda \geq 0$. *Then*

$$w(t;\lambda_1) \geq w(t;\lambda_2) \quad \text{on } [0,1] \quad \text{if } \lambda_1 \geq \lambda_2 \geq 0 \tag{2.5}$$

and

$$w(0;\lambda_1) > w(0;\lambda_2) \quad \text{if } \lambda_1 > \lambda_2 \geq 0. \tag{2.6}$$

Proof. We first prove (2.5). If (2.5) were not true, then $w(t;\lambda_1) \leq w(t;\lambda_2)$ for some $t \in [0,1)$ and there would exist an interval (a,b),

$0 \le a < b \le 1$, such that

$$w(t;\lambda_1) < w(t;\lambda_2) \quad \text{in } (a,b), \qquad w(a;\lambda_1) \le w(a;\lambda_2)$$

and

$$w(b;\lambda_1) = w(b;\lambda_2)$$

by the condition that $w(1;\lambda_1) = w(1;\lambda_2) = 0$.

From this and (2.2), we deduce that

$$w'(t;\lambda_1) - w'(t;\lambda_2) < 0 \quad \text{in } (a,b),$$

which leads to

$$0 = w'(b;\lambda_1) - w'(b;\lambda_2) < w'(a;\lambda_1) - w'(a;\lambda_2) \le 0,$$

a contradiction. (2.5) is thus proved.

We now prove (2.6). It follows from (2.5) that $w(0;\lambda_1) \ge w(0;\lambda_2)$. If $w(0;\lambda_1) = w(0;\lambda_2)$, then

$$w'(0;\lambda_1) - w'(0;\lambda_2) = \frac{\lambda_2}{w(0;\lambda_2)} - \frac{\lambda_1}{w(0;\lambda_1)} < 0.$$

From this, we conclude that there exists a sufficiently small $\delta > 0$ such that

$$w(t;\lambda_1) - w(;\lambda_2) < 0 \quad \text{in } (0,\delta),$$

which contradicts (2.5). This shows that $w(0;\lambda_1) > w(0;\lambda_2)$ when $\lambda_1 > \lambda_2 \ge 0$.

In very much the same way, we can prove the following uniqueness result.

THEOREM 2.3. *Under the assumptions of Theorem* 2.1, *the positive solution* $w(t;\lambda)$ *is unique*.

To establish the existence of the positive solution $w(t;\lambda)$ for each fixed $\lambda \ge 0$, we consider the nonlinear boundary value problem

$$\begin{cases} w''(t) = -\lambda\left(\dfrac{1-t^2}{w(t)}\right)' - \dfrac{t}{w(t)}, & 0 \le t < 1, \\ w'(0)w(0) = -\lambda, & w(1) = \dfrac{1}{j}, \end{cases} \tag*{$(2.7)_j$}$$

for each fixed natural number j. The problem $(2.7)_j$ has no singularity at $t = 1$ and may be viewed as a perturbation of the problem (1.6)–(1.7).

Concerning the problem $(2.7)_j$, we can establish the second comparison principle.

LEMMA 2.4. *Assume that* $\lambda \geq 0$ *and the problem* $(2.7)_j$ *has a positive solution* $w_j(t)$. *Then for all* $j \geq 1$

$$w_j(t) \geq w_{j+1}(t) \geq w(t;\lambda) \quad on \quad [0,1] \tag{2.8}$$

and

$$0 \leq w_j(t) - w_{j+1}(t) \leq \frac{1}{j(j+1)} \quad on \quad [0,1]. \tag{2.9}$$

Proof. The proof of (2.8) is similar to that of (2.5) and hence omitted here. (2.9) follows from (2.8) and (2.1).

Now we are going to establish the existence of the positive solution $w(t;\lambda)$.

THEOREM 2.5. *For each fixed* $\lambda \geq 0$, *the problem* (1.6)–(1.7) *has a* (*unique*) *positive solution* $w(t;\lambda)$.

Proof. Let us define a mapping $\Phi\colon D \mapsto D$ by

$$(\Phi w)(t) := \frac{1}{j} + \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)\,ds}{w(s)} + (1-t)\int_0^t \frac{s\,ds}{w(s)},$$
$$0 \leq t \leq 1,$$

for any $w \in D$, where $D := \{w \in C[0,1];\ w(t) \geq 1/j,\ 0 \leq t \leq 1\}$. From the definition of Φ, we have, for any $w \in D$,

$$(\Phi w)'(t) = -\frac{\lambda(1-t^2)}{w(t)} - \int_0^t \frac{s\,ds}{w(s)}, \qquad 0 \leq t \leq 1,$$

$$|(\Phi w)'(t)| \leq j(\lambda+1), \qquad \frac{1}{j} \leq (\Phi w)(t) \leq j(\lambda+2), \qquad 0 \leq t \leq 1,$$

$$(\Phi w)''(t) = -\lambda\left(\frac{1-t^2}{w(t)}\right)' - \frac{t}{w(t)}, \qquad 0 \leq t \leq 1,$$

and

$$(\Phi w)'(0)w(0) = -\lambda, \qquad (\Phi w)(1) = \frac{1}{j}.$$

This shows that $\Phi(D)$ is a compact subset of D and a fixed point of Φ in D is exactly a positive solution to $(2.7)_j$.

It is easy to check that Φ is a compactly continuous mapping from D into D. The Schauder fixed point theorem tells us that Φ has at least one fixed point in D. Let $w_j(t)$ be a fixed point. Then $w_j(t)$ is a positive solution to $(2.7)_j$,

$$w_j(t) = \frac{1}{j} + \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)}{w_j(s)} ds + (1-t)\int_0^t \frac{s\,ds}{w_j(s)}, \qquad 0 \le t \le 1 \tag{2.10}$$

and $w_j(t) \ge A(1-t)$ on $[0,1]$ (by Lemma 2.4 and Theorem 2.1).

From Lemma 2.4, we come to the conclusion that the sequence $\{w_j(t)\}_{j=1}^{\infty}$ is monotone decreasing and uniformly convergent on $[0,1]$. Let

$$w(t;\lambda) := \lim_{j\to\infty} w_j(t) \quad \text{for } t \text{ in } [0,1].$$

Then $w(t;\lambda) \ge A(1-t)$ on $[0,1]$. Letting $j \to \infty$ in (2.10) and then applying the dominated convergence theorem, we obtain

$$w(t;\lambda) = \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)}{w(s;\lambda)} ds + (1-t)\int_0^t \frac{s\,ds}{w(s;\lambda)}, \qquad 0 \le t \le 1. \tag{2.11}$$

This shows that the $w(t;\lambda)$ is exactly a positive solution to the problem (1.6)–(1.7).

Finally, we prove the following nonexistence statement.

THEOREM 2.6. *If* $\lambda \le -\frac{1}{2}$, *then the problem* (1.6)–(1.7) *has no positive solution.*

Proof. Assume that $w(t)$ is a positive solution to the problem (1.6)–(1.7) with $\lambda \le -\frac{1}{2}$. Then from (2.1) we lead to

$$0 < w(0) = \int_0^1 \frac{(1-s)(\lambda + \lambda s + s)\,ds}{w(s)} < 0,$$

a contradiction. This shows that Theorem 2.6 is true.

3. PROBLEM (1.1)–(1.2)

In this section, we construct the unique solution $f(\eta,\lambda)$ to the problem (1.1)–(1.2) with $\lambda \ge 0$ by utilizing the unique positive solution $w(t;\lambda)$ to the problem (1.6)–(1.7) and describe the asymptotic behavior of $f'(\eta;\lambda)$ as $\eta \to +\infty$.

Concerning the problem (1.1)–(1.2), we can prove the following existence and uniqueness theorem.

THEOREM 3.1. *For each fixed* $\lambda \geq 0$, *the problem* (1.1)–(1.2) *has a unique solution* $f(\eta;\lambda)$, *which satisfies*

$$0 < f'(\eta;\lambda) < 1 \quad \textit{and} \quad f''(\eta;\lambda) > 0 \quad \textit{for all } \eta > 0. \tag{3.1}$$

Proof. Let $\lambda \geq 0$ and let $w(t;\lambda)$ be the unique solution to the problem (1.6)–(1.7). Then the function

$$g(t;\lambda) := \int_0^t \frac{ds}{w(s;\lambda)}, \qquad 0 \leq t < 1.$$

is strictly increasing in $[0,1)$ and $g(0;\lambda) = 0$. From Theorem 2.1, it follows that $g(1-0;\lambda) = +\infty$ and also $g'(1-0;\lambda) = +\infty$. Let $t = h(\eta;\lambda)$ be the inverse function to $\eta = g(t;\lambda)$ and let

$$f(\eta;\lambda) := \int_0^\eta h(s;\lambda)\,ds, \qquad 0 \leq \eta < +\infty. \tag{3.2}$$

Then $f'(\eta;\lambda) \equiv h(\eta;\lambda)$ on $[0,+\infty)$,

$$f(0;\lambda) = 0, \qquad f'(0;\lambda) = 0, \qquad f'(+\infty;\lambda) = 1 \tag{3.3}$$

and

$$\eta \equiv g(f'(\eta;\lambda);\lambda) = \int_0^{f'(\eta;\lambda)} \frac{ds}{w(s;\lambda)} \quad \text{for all } \eta \geq 0. \tag{3.4}$$

We are going to prove that the $f(\eta;\lambda)$ is a solution to (1.1)–(1.2).

Differentiating (3.4) with respect to η yields

$$f''(\eta;\lambda) = w(f'(\eta;\lambda);\lambda) > 0 \quad \text{for all } \eta \geq 0. \tag{3.5}$$

Inserting $t = f'(\eta;\lambda)$ into (2.2) and then using (3.5), we obtain

$$\begin{aligned} w'(f'(\eta;\lambda);\lambda) &= -\int_0^{f'(\eta;\lambda)} \frac{s\,ds}{w(s;\lambda)} - \frac{\lambda\left[1-(f'(\eta;\lambda))^2\right]}{w(f'(\eta;\lambda);\lambda)} \\ &= -f(\eta;\lambda) - \frac{\lambda\left[1-(f'(\eta;\lambda))^2\right]}{f''(\eta;\lambda)}, \qquad 0 \leq \eta < +\infty. \end{aligned}$$

Differentiating (3.5) with respect to η and then using the above, we get

$$\begin{aligned} f'''(\eta;\lambda) &= w'(f'(\eta;\lambda);\lambda) f''(\eta;\lambda) \\ &= -f(\eta;\lambda) f''(\eta;\lambda) - \lambda\left[1 - (f'(\eta;\lambda))^2\right], \qquad 0 \le \eta < +\infty. \end{aligned} \tag{3.6}$$

(3.6) along with (3.3) shows that the function $f(\eta;\lambda)$ defined by (3.2) is exactly a solution to (1.1)–(1.2). It is clear that (3.1) is true.

The uniqueness of $f(\eta;\lambda)$ follows from that of $w(t;\lambda)$. Up to now Theorem 3.1 is proved.

So far as the problem (1.1)–(1.2) is concerned, $f' = u/U$ exhibits the velocity distribution in the laminar boundary layer in the flow past a wedge given by $U(x) = Bx^m$, $B > 0$, $m := \lambda/(2-\lambda)$ (see [1, p. 292; 7, p. 151]), so people are not interested in f but in f'.

Concerning $f'(\eta;\lambda)$, the following comparison principle holds.

THEOREM 3.2. *Let* $f(\eta;\lambda)$ *be the unique solution to the problem* (1.1)–(1.2) *with* $\lambda \ge 0$. *Then*

$$f'(\eta;\lambda_1) > f'(\eta;\lambda_2) \quad \textit{for all } \eta > 0 \ \textit{ if } \ \lambda_1 > \lambda_2 \ge 0. \tag{3.7}$$

Proof. According to (3.4), we have

$$\eta = \int_0^{f'(\eta;\lambda_1)} \frac{ds}{w(s;\lambda_1)} = \int_0^{f'(\eta;\lambda_2)} \frac{ds}{w(s;\lambda_2)} \quad \text{for all } \eta \ge 0.$$

Adapting the above in the following form and then applying Theorem 2.2, we obtain

$$\int_{f'(\eta;\lambda_2)}^{f'(\eta;\lambda_1)} \frac{ds}{w(s;\lambda_1)} = \int_0^{f'(\eta;\lambda_2)} \left(\frac{1}{w(s;\lambda_2)} - \frac{1}{w(s;\lambda_1)} \right) ds > 0, \qquad \eta > 0,$$

which implies (3.7).

Finally, we describe the asymptotic behavior of $f'(\eta;\lambda)$ as $\eta \to \infty$.

THEOREM 3.3. *Let* $f(\eta;\lambda)$ *be the unique solution to the problem* (1.1)–(1.2) *with* $\lambda \ge 0$. *Then*

$$\exp\{-(2\lambda+1)(e^{\eta/A} - 1)\} \le 1 - f'(\eta;\lambda) \le e^{-A\eta} \quad \textit{for all } \eta \ge 0,$$

where $A := \sqrt{(4\lambda+1)/6}$.

Proof. From (2.3), we know that

$$g(t;\lambda) = \int_0^t \frac{ds}{w(s;\lambda)} \le \int_0^t \frac{ds}{A(1-s)}$$
$$= -\frac{1}{A}\log(1-t), \qquad 0 \le t < 1.$$

Inserting $t = f'(\eta;\lambda)$ into the above gives

$$\eta \le -\frac{1}{A}\log(1 - f'(\eta;\lambda)) \quad \text{for all } \eta \ge 0,$$
$$1 - f'(\eta;\lambda) \le e^{-A\eta} \quad \text{for all } \eta \ge 0.$$

Similarly, from (2.4), we obtain

$$g(t;\lambda) \ge A\int_0^t \frac{ds}{(1-s)[2\lambda + 1 - \log(1-s)]}$$
$$= A\log\left[1 - \frac{\log(1-t)}{2\lambda+1}\right], \qquad 0 \le t < 1.$$

Substituting $t = f'(\eta;\lambda)$ into the above yields

$$\eta \ge A\log\left[1 - \frac{\log(1 - f'(\eta;\lambda))}{2\lambda+1}\right] \quad \text{for all } \eta \ge 0.$$

From this we deduce that

$$1 - f'(\eta;\lambda) \ge \exp\{-(2\lambda+1)(e^{\eta/A}\text{-}1)\} \quad \text{for all } \eta \ge 0.$$

This concludes the proof of Theorem 3.3.

The following nonexistence statement is a direct consequence of Theorem 2.6.

THEOREM 3.4. *For $\lambda \le -\frac{1}{2}$, the boundary value problem* (1.1)–(1.2) *has no solution satisfying* (3.1).

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